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LETTER TO THE EDITOR

On the critical region of systems with two order parameters

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Abstract. The critical region in the disordered phase near bicritical and tetracritical points is obtained. For this purpose the condition for the validity of the Ornstein–Zernike approximation for the correlation functions is used. The consideration holds for $2<d<4$ dimensions of space. The effect of the interaction between the two orderings near the multicritical points is discussed.

Recently, the systems with two second-order phase transitions where bicritical and tetracritical points appear are of current experimental and theoretical interest. The intrinsic features of these multicritical points are contained in the following Ginzburg–Landau (GL) Hamiltonian

$$\mathcal{H}=-\frac{1}{2}\int \mathrm{d}^d x\{a_1^0\varphi_1^2+c_1(\nabla\varphi_1)^2+2u_1\varphi_1^4+a_2^0\varphi_2^2+c_2(\nabla\varphi_2)^2+2u_2\varphi_2^4+4w\varphi_1^2\varphi_2^2\} \tag{1}$$

where the factor $(-1/K_\mathrm{B}T)$ is absorbed in $\mathcal{H}$. The order parameters $\varphi_i(\boldsymbol{x})(i=1,2$ hereafter) depends on the coordinate $\boldsymbol{x}$ in the d-dimensional space. The order parameters $\varphi_1(\boldsymbol{x})$ and $\varphi_2(\boldsymbol{x})$ are assumed to be n_1- and n_2-component real fields, respectively. The parameters in (1) are analytic functions of the thermodynamic variables—the temperature T and the additional ones (like pressure etc) which we shall denote by ζ. As usual, $a_i^0=\alpha_i(T-T_{ci}^0)$, the superscript zero denoting the bare values of the critical temperatures $T_{ci}^0(\equiv T_{ci}^0(\zeta))$ as other bare quantities here. For the renormalised quantities the superscript zero will be omitted.

The simplest theoretical analysis of the Hamiltonian (1) is performed in the mean-field approximation that is widely discussed by Liu and Fisher (1973) and by Imry (1975). The Wilson renormalisation group analysis of this Hamiltonian is also developed (e.g. Kosterlitz *et al* 1976, see also Patashinsky and Pokrovsky 1977a, b and the references therein). The Ornstein–Zernike (OZ) approximation (often refered to as the Gaussian approximation) is also used when some non-universal properties of the fields φ_i are studied.

The Hamiltonian (1) describes a large number of real systems. Many of them are listed and studied in the papers mentioned above. Further examples are, for instance, the superconducting compounds with long-range magnetic orderings, say, rare-earth molybdenum chalcogenides with the formula RE Mo_6X_8 (RE being rare-earth ion(s) and X = S, Se; see Fisher *et al* (1975) and Shelton *et al* (1976)) or rare-earth rhodium borides RE Rh_4B_4 (Mattias *et al* 1977). For the theoretical investigation of these compounds microscopic models are used, where fluctuations near the superconducting phase transition and far from the multicritical points are neglected (e.g. Maekawa and

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Tachiki 1978, Youngner and Machida 1979, Machida 1979). In a recent paper, using Hamiltonian (1) for three-dimensional ferromagnetic superconductors Hornreich and Schuster (1979) point out a great enlargement of the critical region for the superconducting order parameter (say φ_1). This enlargement makes it possible to look for an experimental observation of the three-dimensional superconducting fluctuations, for instance, in the superconducting compounds mentioned above. Then the question of determining the size of the critical region near multicritical points arises. In the disordered phase this problem reduces to an investigation of the applicability of the OZ approximation near bicritical and tetracritical points.

The critical region of an ordinary second-order phase transition is well known (Levanyuk 1959, Ginzburg 1960a, b). This subject is discussed in many other papers (see Amit 1974 and the references therein).

Here we estimate the critical region in the disordered phase near the multicritical points where the OZ approximation for the correlation functions of Hamiltonian (1) breaks down. We shall discuss the critical region near the shifted (renormalised) bicritical and/or tetracritical points (T_c, ζ_c) and near the transition lines $T_{ci}(\zeta)$. The consideration is referred to $2<d<4$ dimensions of space. The Hamiltonian (1) has a symmetry with respect to the change of the suffices 1 and 2. Then the results connected with the first phase transition ($i=1$) correspond to the second phase transition ($i=2$) after the change ($1 \rightleftarrows 2$).

For our purposes we consider Dyson's equation for the correlation function $\chi_1(\boldsymbol{q})$ of the field $\varphi_1(\boldsymbol{q})$ in momentum space, namely,

$$\chi_1^{-1}(\boldsymbol{q}) = c_1\boldsymbol{q}^2 + \alpha_1(T - T_{c_1}^0) - \Sigma_1(\boldsymbol{q}, T) \tag{2}$$

where $\Sigma_1(\boldsymbol{q}, T)$ is the self-energy function. The reverse correlation function $\chi_1^{-1}(0)$ is equal to zero at the (true) renormalised transition line $T_{c_1}(\zeta)$ separating the disordered phase from the ordered one. From equation (2) it follows that

$$\alpha_1(T_{c_1} - T_{c_1}^0) - \Sigma_1(0, T_{c1}) = 0 \tag{3}$$

on the transition line $T_{c_1}(\zeta)$. Subtraction of equation (3) from equation (2) leads to the expression

$$\chi_1^{-1}(\boldsymbol{q}) = \chi_1^{(0)-1}(\boldsymbol{q}) - \Delta\Sigma_1(\boldsymbol{q}, T) \tag{4}$$

where

$$\chi_i^{(0)}(\boldsymbol{q}) = (a_i + c_i\boldsymbol{q}^2)^{-1} \qquad (a_i = \alpha_i(T - T_{ci})) \tag{5}$$

is the OZ approximation for the correlation functions $\chi_i(\boldsymbol{q})$ with the renormalised critical temperatures T_{ci},

$$\Delta\Sigma_1(\boldsymbol{q}, T) = \Delta\Sigma_1(T) + \Delta\Sigma_1(\boldsymbol{q}) \tag{6}$$

with

$$\Delta\Sigma_1(T) = \Sigma_1(0, T) - \Sigma_1(0, T_{c1}) \tag{7}$$

and

$$\Delta\Sigma_1(\boldsymbol{q}) = \Sigma_1(\boldsymbol{q}, T) - \Sigma_1(0, T). \tag{8}$$

For the OZ form (6) to be valid one has to require

$$ci\boldsymbol{q}^2 \gg |\Delta\Sigma_1(\boldsymbol{q})| \tag{9}$$

and

$$a_1 \gg |\Delta\Sigma_1(T)|. \tag{10}$$

The inequalities (9) and (10) are common criteria for the validity of the OZ approximation for the correlation function χ_1 of any GL Hamiltonian. The quantity $\Delta\Sigma_1(\boldsymbol{q})$ in (9) is of order $(u_i^2, w^2, u_i w)$ while $\Delta\Sigma_1(T)$ in (10) is of order (u_i, w). As in the derivation of the Ginzburg–Levanyuk criterion for the one-field GL Hamiltonian (see e.g. Amit 1974) here we shall consider only the first-order contributions in $\Delta\Sigma_1(\boldsymbol{q}, T)$.

In the framework of the model (1) the following expression for $\Sigma_1(T)$ is obtained

$$\Sigma_1^{(1)}(T) = -4(n_1+2)u_1 J(a_1) - 4n_2 w J(a_2) \tag{11}$$

where

$$J(a_i) = \int_0^\Lambda \mathrm{d}\boldsymbol{q}(c_i\boldsymbol{q}^2 + a_i)^{-1} \tag{12}$$

($\int \mathrm{d}\boldsymbol{q} = (2\pi)^{-d}\int \mathrm{d}^d q$, Λ is the momentum cut-off).

Below, without loss of generality, we shall assume $T_{c_1} \geqslant T_{c_2}$. Using (11)–(12) we obtain for equation (3) the expression

$$T_{c_1} = T_{c_1}^0 - \frac{2A(d)}{\pi(d-2)}\Lambda^{d-2}\xi_{01}^2\left((n_1+2)\frac{u_1}{c_1^2} + n_2\frac{w}{c_1c_2}\right)$$
$$+ A(d)\frac{n_2 w}{c_1 c_2}\xi_{01}^2\xi_{02}^{2-d}(T_{c_1} - T_{c_2})^{(2-d)/2} \tag{13}$$

where $A(d) = 2^{2-d}\pi^{1-d/2}\Gamma^{-1}(\frac{1}{2}d)$, $\Gamma(z)$ is the gamma function and $\xi_{0i}/(2T_{ci})^{1/2} = (c_i/2T_{ci}\alpha_i)^{1/2}$ are the zeroth ($T=0$) correlation lengths of the decoupled ($w=0$) system (1). The last term in (13) vanishes in the multicritical point (T_c, ζ_c) where $T_{c_1} = T_{c_2}(\equiv T_c)$. Equation (13) for T_{c1} gives a straightforward generalisation of the well known shift $(T_{c_1}^0 - T_{c_1})$ obtained by Vaks *et al* (1967). For an explicit expression for T_{c_1} (in particular for T_c) to be obtained one has to determine the parameters of Hamiltonian (1) as functions of T and ζ. This may be achieved when a microscopic analysis of the considered system is carried out.

From (11)–(12) we obtain a concretisation of the inequality (10):

$$(T - T_{c_1})^{(4-d)/2} \gg \Delta_{u_1}\left\{1 + \frac{\Delta_w}{\Delta_{u_1}}\left[\left(1 + \frac{T_{c_1} - T_{c_2}}{T - T_{c_1}}\right)^{(d-2)/2} - \left(\frac{T_{c_1} - T_{c_2}}{T - T_{c_1}}\right)^{(d-2)/2}\right]\right\} \tag{14}$$

where

$$\Delta_{u_1} = \frac{A(d)(n_1+2)u_1}{\alpha_1^2\xi_{01}^d} \tag{15}$$

$$\Delta_w = \frac{A(d)n_2 w}{\alpha_1\alpha_2\xi_{02}^d}. \tag{16}$$

The criterion (14) for the validity of the OZ approximation is not an explicit expression for $(T - T_{c_1})$. It is seen from equations (14)–(16) that for $w = 0$ equation (14) is reduced to the well known Ginzburg–Levanyuk criterion. Then the critical region is determined by the inequality $(T - T_{c_1})^{(u-d)/2} \leqslant \Delta_{u_1}$. We shall briefly comment on three particular cases that recover the most essential features of criterion (14):

(i) Far from the multicritical point, i.e. $(T_{c_1}-T_{c_2}) \gg (T-T_{c_1})$. Then from inequality (14) it is easy to obtain for $(T-T_{c_1})$

$$(T_{c_1}-T_{c_2})^{(4-d)/2} \gg (T-T_{c_1})^{(4-d)/2} \gg \frac{A(d)(n_1+2)}{\alpha_1^2 \xi_{01}^d} \tilde{u}_1 \tag{17}$$

where

$$\tilde{u}_1 = \frac{u_1}{1-\frac{1}{2}(d-2)\Delta_w (T_{c_1}-T_{c_2})^{(d-4)/2}}. \tag{18}$$

Expression (17) is a Ginzburg–Levanyuk form with a renormalised coupling constant $\tilde{u}_1$ (according to (18)). The expression for $\tilde{u}_1$ is valid if the second term in the denominator is less than unity. From equations (17)–(18) we see that the fluctuations φ_1 are already perturbed by the coupling w and the critical region of the first ($i=1$) phase transition increases.

(ii) Near the multicritical point, i.e. $(T_{c1}-T_{c2}) \ll (T-T_{c1})$. In this case we present the explicit expression for $(T-T_{c1})$ in three dimensions ($d=3$) of space

$$(T-T_{c1})^{1/2} > (\Delta_{u_1}+\Delta_w)\left(1-\frac{\Delta w}{(\Delta u_1+\Delta_w)^2}(T_{c_1}-T_{c_2})^{1/2}\right). \tag{19}$$

The inequality (19) is valid if the second term in the large brackets is less than unity, namely, for sufficiently small values of $(T_{c_1}-T_{c_2})$.

Now we assume the critical region of the second phase transition ($i=2$) to be wider than that of the first phase transition ($i=1$), namely, we shall consider the case $\xi_{01} \gg \xi_{02}$. Then we find from (19)

$$(T-T_{c_1})^{1/2} \gg \frac{A(3)n_2}{\xi_{02}^3 \alpha_1 \alpha_2} w - (T_{c_1}-T_{c_2})^{1/2}. \tag{20}$$

In this case as it follows from (20) the critical region of the first phase transition depends on the correlation length ξ_{02} instead of ξ_{01}. As a result of the inequality ξ_{01} ξ_{02} it increases. Let us now compare the sizes of the critical regions: $(T-T_{c_1})_w$, for the field φ_1, $(T-T_{c_1})_{w=0}$, for the field φ_1 when $w=0$ and $(T-T_{c_2})_{w=0}$, the critical region for the field φ_2 in the case $w=0$. Note, that: $(T-T_{c_1})_{w=0} \simeq \Delta_{u_1}$ and $(T-T_{c_2})_{w=0} \simeq \Delta_{u_2}$, where Δ_{u_2} is obtained by the change $(1 \rightleftarrows 2)$ in (15). Using (19) and (20) we have

$$(T-T_{c_1})_w^{1/2} \simeq \frac{n_2}{n_1+2}\frac{\alpha_1 w}{\alpha_2 u_1}\left(\frac{\xi_{01}}{\xi_{02}}\right)^3 (T-T_{c_1})_{w=0}^{1/2} - (T_{c_1}-T_{c_2})^{1/2} \tag{21}$$

and

$$(T-T_{c_1})_w^{1/2} \simeq \frac{n_2}{n_2+2}\left(\frac{w\alpha_2}{u_2\alpha_1}\right)(T-T_{c_2})_{w=0}^{1/2} - (T_{c_1}-T_{c_2})^{1/2}. \tag{22}$$

From (21) it is obvious that the critical region of φ_1 is greatly enlarged ($\xi_{01} \gg \xi_{02}$). Assuming $\alpha_2 w \sim \alpha_1 u_2$ we see from (22) that the critical region $(T-T_{c_1})_w$ becomes comparable with $(T-T_{c_2})_{w=0}$. This result is obtained by Hornreich and Schouster (1979) for $T_{c_1}=T_{c_2}=T_c$. If we assume, in addition to the inequality $\xi_{01} > \xi_{02}$, that we have the particular case $u_1 \sim u_2 < w$ of the bicritical behaviour then it follows from (22) that the critical region $(T-T_{c_1})_w$ may become wider than that of the field φ_2. In comparison with the case (1) the influence of the quantity $(T_{c_1}-T_{c_2})$ here (see equations (21) and (22)) is in the opposite direction, i.e. it suppresses the effect of the enlargement.

(iii) On the multicritical point, i.e. $T_{c_1} = T_{c_2} = T_c$. First of all the expression for T_c can be obtained both from equation (13) and from the analogous one for T_{c_2} which follows from (13) after the change $(1 \rightleftarrows 2)$. Then in addition to the expression for T_c we obtain the following constraint for the parameters u_i, w, c_i and α_i.

$$\xi_{01}^2\left((n_1+2)\frac{u_1}{c_1^2}+n_2\frac{w}{c_1c_2}\right)=\xi_{02}^2\left((n_2+2)\frac{u_2}{c_2^2}+n_1\frac{w}{c_1c_2}\right). \tag{23}$$

Using equation (23) one may find the coordinate ξ_c of the multicritical point (T_c, ζ_c). For this purpose the explicit dependence of u_i, w, c_i and α_i on ζ is necessary.

Using (14) and the corresponding inequality for $(T-T_{c_2})$ we obtain an estimate for the critical region around T_c, namely,

$$(T-T_c)^{(4-d)/2} \lessapprox \max\{\Delta_{u_1}+\Delta_w, \Delta_{u_2}+\Delta_w\}. \tag{24}$$

For $u_1 = u_2 = w$, $c_1 = c_2$ and $\alpha_1 = \alpha_2$, the inequality (24) defines the critical region of an ordinary second-order phase transition. If $\xi_{01} \gg \xi_{02}$, from (24) we have

$$(T-T_c)^{(4-d)/2} \lessapprox \frac{A(d)}{\xi_{02}^{\alpha}\alpha_2}\max\left\{\frac{n_2}{\alpha_1}w, \frac{n_2+2}{\alpha_2}u_2\right\}$$

for the critical region.

Note that for $T_{c_1} = T_{c_2} = T_c$, the corresponding expressions (21) and (22) may be written in $2 < d < 4$ dimensions of space.

From an experimental point of view the possibility for a great enlargement of the critical region of a second-order phase transition near bicritical and tetracritical point is of special interest. This effect is maximally pronounced when the temperature of the two phase transitions coincide (case (iii)). However this case is extremely rarely realised in nature. Usually T_{c_1} and T_{c_2} are different. We have shown that the enlargement of the critical region takes place also when $T_{c_1} \neq T_{c_2}$ (case (2)) but it becomes negligible as the difference between T_{c_1} and T_{c_2} increases (case (i)).

Here we have presented a part of the problem for determination of criteria for the validity of the mean-field and the OZ approximations near bicritical and tetracritical points. The following should be noted.

(i) The problem is not restricted only in the framework of the Hamiltonian (1). Other GL forms describing bicritical and tetracritical points are also possible. Such as, for example, the GL forms for the exitonic superconductor (see Kopayev and Molotkov 1979) and for the exitonic ferromagnet in a magnetic field (Volkov *et al* 1980).

(ii) A full discussion of the problem requires determination of the critical regions for other quantities (e.g. specific heat, etc) and an investigation of the ordered phases too.

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